Two-state on-off intermittency and the onset of turbulence in a spatiotemporally chaotic system

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We show the existence of two-state on-off intermittent behavior in spatially extended dynamical systems, using as an example the damped and forced drift wave equation. The two states are stationary solutions corresponding to different wave energies. In the language of (Fourier mode) phase space these states are embedded in two invariant manifolds that become transversely unstable in the regime where two-state on-off intermittency sets in. The distribution of laminar duration sizes is compatible with the similar phenomenon occurring in time only in the presence of noise. In extended system the noisy effect is provided by the spatial modes excited by the perturbation. We show that this intermittency is a precursor of the onset of strong turbulence in the system.

The onset of turbulence in spatiotemporal systems is a long-standing problem of paramount importance, which has been intensively studied in the past decades [1]. If we consider a Fourier mode description of the transition to turbulence in a spatially extended dynamical system, the onset of turbulence occurs when the system energy, originally concentrated in the temporal degree of freedom, is distributed among the spatial modes. Moreover, the temporal dynamics must present a chaotic attractor acting as a stochastic pump, by feeding energy to the spatial modes to be excited in order to yield irregular spatial behavior [2, 3]. In a previous paper, we described this process in a system of three waves nonlinearly coupled through interactions of a wave triplet subjected to a resonance condition [4]. The temporal dynamics, exhibiting chaotic behavior but a periodic spatial profile, can be viewed geometrically as a low-dimensional manifold embedded in the phase space consisting of the Fourier modes retained in the procedure to solve the system. The onset of turbulence occurs when this manifold loses transversal stability such that the trajectories are allowed to explore more spatial degrees of freedom, imparting energy to the corresponding spatial modes [5].

In this letter we consider a situation where there are two manifolds instead of only one, in each of which lie stationary solutions of a periodically forced nonlinear wave equation corresponding to different wave energies. These manifolds lose transversal stability through the same process as before, but there are novel features in this case. One of the most important is that both manifolds may be simultaneously unstable in the transversal direction to them. In this case the trajectories wander through the available phase space volume approaching the vicinity of both manifolds in an erratic way. This behavior, also known as two-state on-off intermittency, has been described in low-dimensional dynamical systems [8] as well as in spatially extended systems [9]. In this kind of intermittency there is an alternating behavior between two different stationary states. Due to the chaotic behavior in each state, these alternations occur for different and irregularly spaced time intervals. Two-state intermittency differs from conventional on-off intermittency [10] because the system state, after departing from a given state, goes to the vicinity of another state, the transient behavior in between these states being governed by a chaotic saddle [11]. In spatially extended systems the dynamics at either state lies on a manifold embedded in the phase space.

As a representative example of a spatially extended dynamical system with two solution branches we consider the damped and forced drift wave equation \[ \phi_t + a \phi_{xx} + c \phi + f \phi_x + \gamma \phi = -\epsilon \sin(K x - \Omega t), \]

For magnetically confined fusion plasmas \( \phi(x,t) \) is the non-dimensional electric potential of a drift wave propagating along the poloidal direction of a toroidal plasma, where the constants \( a, c, \) and \( f \) stand for plasma and wave parameters, and we introduced a phenomenological linear damping term with coefficient \( \gamma \) [13]. The effect of other possibly relevant modes is represented by a time-periodic driving with amplitude \( \epsilon \), wave number \( K \) and frequency \( \Omega \).

Equation (1), for certain parameters values, describes a transition from pure temporal chaos without spatial mode excitation to spatio-temporal chaos. This transition has been described in Refs. [6, 7] as resulting from an interior crisis, whereby a chaotic attractor collides with a high-dimensional chaotic saddle. However, the intermittent behavior that follows from an interior crisis turns to be different from that observed in the numerical simulations. We found that the solutions wander in an intermittent fashion between two \textit{non-overlapping} states of distinct wave energies, whereas an interior crisis would yield an intermittent alternation between overlapping states: one of which is the pre-critical attractor “ghost”, and the other is the post-critical attractor made available by the crisis. Since these states are non-overlapping, we propose that the observed behavior is due not to an interior crisis but rather due to two-state on-off intermittency.

Since the \( x \)-coordinate is either an angle or can be a bounded variable, we suppose a Fourier expansion

\[ \phi(x,t) = \sum_{n=0}^{N} |\varphi_n(t)| e^{i n x}, \]

where the \( \varphi_n(t) \) are the amplitudes of the \( n \)-th Fourier mode. The \( \varphi_n(t) \) are determined by the following system of coupled nonlinear differential equations:

\[ \begin{align*}
\varphi_{nn} + J_n &\left( \varphi_n \right) + \sum_{m} M_{nm} \varphi_m &\quad n \neq 0, \\
\varphi_0 &\quad n = 0,
\end{align*} \]

with the boundary conditions \( \varphi_{n}(-\pi) = \varphi_{n}(\pi) = 0 \) for \( n \neq 0 \). The \( M_{nm} \) are the interactions between the Fourier modes, \( J_n \) is the external forcing, and \( \varphi_0(t) \) is the initial condition. The \( \varphi_n(t) \) are determined by the following system of coupled nonlinear differential equations:

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In conclusion, we have shown the existence of two-state on-off intermittent behavior in spatially extended dynamical systems, using as an example the damped and forced drift wave equation. The two states are stationary solutions corresponding to different wave energies. In the language of (Fourier mode) phase space these states are embedded in two invariant manifolds that become transversely unstable in the regime where two-state on-off intermittency sets in. The distribution of laminar duration sizes is compatible with the similar phenomenon occurring in time only in the presence of noise. In extended system the noisy effect is provided by the spatial modes excited by the perturbation. We show that this intermittency is a precursor of the onset of strong turbulence in the system.
where \( \phi_k(t) \) are time-dependent amplitudes and \( \kappa_n \equiv 2\pi n/L \) in a box of length \( L = 2\pi \) and periodic boundary conditions. Notice that the mode \( \kappa_0 = 0 \) is purely temporal, whereas \( \kappa_n = \alpha = 1, 2, 3, \ldots \) stand for the spatial modes. On substituting (2) into (1), one obtains a system of \( N \) coupled ordinary differential equations, solved using a 12th order Adams predictor-corrector scheme. In the numerical simulations to be reported in this letter we used \( N = 128 \) modes, unless stated otherwise, and we kept fixed the following parameters [13]: \( \alpha = -0.28711, \epsilon = 1.0, f = -6.0, \gamma = 0.1, K = 1.0, \) and \( \Omega = 0.65 \), such that \( \epsilon \) will be our control parameter. The initial conditions for the system of \( N \) coupled mode equations are \( \varphi(0) = 0.01, \varphi_1(0) = \varphi_2(0) = \sigma_1R(0,1), \varphi_n(0) = \sigma_2R(0,1), \) for \( n \geq 3 \), where \( \sigma_1 = 0.001, \sigma_2 = 10^{-5}, \) and \( R(0,1) \) is a pseudo-random number chosen within the interval \([0,1]\) with uniform probability. We stress that these initial conditions are different from those used in Ref. [13], where a solitary-wave-solution of the unperturbed case (\( \epsilon = 0, \gamma = 0 \)) was chosen.

The possible solutions of the initial and boundary value problem defined by Eq. (1) can be best characterized by computing the wave energy, defined as

\[
E(t) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \left[ \phi^2(x,t) - a\phi_x^2(x,t) \right] dx, \tag{3}
\]

which turns to be an integral of motion for the unperturbed case. For the parameter values explored in this work the wave dynamics is chaotic, but even so the energy difference is bounded [6, 14].

In Figure 1 we exhibit the time evolution of the energy difference \( \Delta E = E(t) - E(0) \) for selected values of the control parameter \( \epsilon \). As the latter is increased from zero, we have a steady state energy with a few peaks, and asymptoting to a value about 0.05 [Fig. 1(a)], until a first bifurcation value \( \epsilon_B = 0.199553 \) is achieved. For \( \epsilon < \epsilon_B \leq \epsilon_f \) we have alternation of energy values between two values, the former \( \approx 0.05 \) lower branch and a \( \approx 0.25 \) higher branch (Fig. 1(b)]. Finally, for \( \epsilon > \epsilon_f \) the energy fluctuates about the higher branch value, never to return to the lower branch (Fig. 1(c)).

Our main point in this letter is that the lower and upper branches of the wave energy are two energy states for which, when \( \epsilon_f < \epsilon < \epsilon_B \), there is intermittent behavior. These two states can be represented in the Fourier mode space, by different energy hypersurfaces we call A and B. For \( \epsilon < \epsilon_f \) the state A is stable with respect to transversal displacements from the energy hypersurface, whereas B is transversely unstable and not reached by typical initial conditions (Fig. 1(a)).

Figure 2 shows a schematic picture of the situation, in which the states A and B have average energies of 0.05 and 0.25, respectively. The shaded regions represent the maximum fluctuations of the energy about these states, in order to emphasize that there is no overlapping between them. The state A loses transversal stability at \( \epsilon = \epsilon_f \) and B becomes transversely stable at \( \epsilon = \epsilon_B \). The existence of these two bifurcation points is long known and has been related to hysteretical behavior when increasing and/or decreasing \( \epsilon \) through these critical values [13]. However, since the energy excursions are bounded and non-overlapping, there cannot be such hysteretical behavior, since none of these states are transversely stable. Another observable consequence of the transversal stability properties of the states A and B is that, for \( \epsilon_f < \epsilon < \epsilon_B \), both manifolds are transversely unstable, and the wave energy makes intermittent transitions between these two states (Fig. 1(b)). Finally, for \( \epsilon > \epsilon_B \) only the state B is transversely stable and two-state intermittency ceases (as in Fig. 1(c)).

Since a few temporal modes are excited in the state A, it corresponds to temporal chaos combined with regular (periodic) spatial patterns. This is illustrated by the space-time plot depicted in Fig. 3(a), obtained for \( \epsilon < \epsilon_f \), which displays a chaotic time evolution with regular spatial behavior akin to a travelling wave solution. However, as the state B becomes transversely stable, a large number of spatial modes are excited. A rather extreme example, considering \( \epsilon > \epsilon_B \), is shown in Fig. 3(b), for which we see aperiodic behavior in both spatial and temporal scales (spatio-temporal chaos). For a small interval of time (circa 25 pseudo-periods) some travelling waves, which appear due to the inductor term, are con-
In order to provide a quantitative characterization of the dynamics in space and time we can resort to Lyapunov exponent computation in Fourier space. In this case, each Fourier mode in the discrete transform (2) can be considered a degree of freedom, and the corresponding Lyapunov exponent is computed from the WSVS algorithm [15], with the dynamics given by the set of ordinary differential equations for the wave amplitudes \( \varphi_n(t) \), with \( n = 0, 1, 2, \ldots, N \). The exponent related to \( n = 0 \) corresponds to the temporal dynamics, whereas the \( n \geq 1 \) case stands for spatial degrees of freedom and can be used to detect spatial mode excitation [4].

Accordingly, in Fig. 4 we plot the time evolution of the 30 first Lyapunov exponents out of \( N = 128 \) modes corresponding to the wave amplitudes in Eq. (2). The symbols \( \lambda_L \) and \( \lambda_T \) stand for longitudinal and transversal exponents, respectively. When a given exponent decays with time as a power-law, it is considered zero when time tends to infinity; and, if this decay is faster than a power law, the exponent goes to negative values. In the case \( \epsilon = 0.195 < \epsilon_t \), only the Lyapunov exponent related to the time \( (n = 0) \) asymptotes to a nonzero value [Fig. 4(a)], confirming our claim that only temporal chaos is observed. The exponents corresponding to spatial modes are shown to decay in a roughly power-law \( (n = 1) \) and even faster rates (for \( n \geq 2 \)). Hence those spatial modes, if excited at all, can have at most periodic behavior (and a possible quasiperiodic one). By way of contrast, for \( \epsilon > \epsilon_t \) a large number of the exponents for \( n \neq 0 \) do not vanish, hence many spatial modes become chaotic [Fig. 4(b)]. This spatial mode excitation involving so many Fourier modes suggests the existence of a strongly turbulent state.

The existence of strong turbulence can actually be tested by computing the Fourier spectrum of the waves \(|\varphi_n| = (1/t_{max})\sum_{t=0}^{t_{max}} |\varphi_n(t)|\), where \( \kappa_n = n \). When there is temporal chaos only [i.e., for \( \epsilon < \epsilon_t \)], cf. Fig. 5(a) we have an energy spectrum that decays faster than a power law, starting from a maximum value at \( k = 3 \). A least squares fit gives an exponential scaling \(|\varphi| \sim e^{-\sigma k}\), with \( \sigma = 0.737 \pm 0.014 \). However, in the upper energy branch \( B \), where we believe that a strong turbulent state sets in, the computed Fourier spectrum can be fitted by a power law of the form \(|\varphi| \sim k^{-\bar{\sigma}}\), where \( \bar{\sigma} = 1.558 \pm 0.019 \) [Fig. 5(b)]. In the latter figure we have used \( N = 1024 \) modes since, in the fully turbulent case, where the strong interaction there existing among different spatial scales leads to a fast redistribution of the wave energy to the lowest wavelengths, such that the use of a large number \( N \) of modes is mandatory.

Our numerical evidences of strong turbulent behavior in the upper energy branch \( B \) are based on the Taylor hypothesis, namely we can relate the spatial statistics corresponding to a fixed time to the statistics of a time series measured at a single point of space [16]. Hence we may characterize the turbulent behavior by the energy spectrum as a function of the frequency rather than the wave number. The energy spectrum \( E(\nu) \) corresponding to the Fourier spectra we considered above is depicted in Fig. 5(c), where we superimposed a straight line corresponding to the Kolmogorov scaling \( E(\nu) \sim \nu^{-5/3} \) to guide the eye. We see basically three frequency intervals with respect to the Kolmogorov scaling: (i) for small fre-
Two-state on-off intermittency occurs in the case $\epsilon_\ell < \epsilon < \epsilon_h$ and is characterized by the irregular alternations between the two energy branches $A$ and $B$, which are transversely unstable in the Fourier-mode phase space of the system. If the trajectory starts at a point close (off but very near) to $A$ the system stays for some time at the lower energy branch, until it approaches a transversely unstable periodic orbit embedded in the invariant manifold $A$ and is ejected away from $A$ towards $B$, essentially the same behavior occurring there. By adopting a small tolerance in the vicinity of each invariant manifold we can define “laminar” states - or plateaus - as those in which the dynamics is very near either of the invariant manifolds. The duration of these plateaus are rather arbitrary since the dynamics is temporarily chaotic and the system ergodically approaches every accessible transversely unstable orbit in both manifolds. Hence, a statistical characterization of the two-state on-off intermittency can be given by the probability distribution function of the plateau sizes (or laminar durations) $\tau_i$ in the intermittent regime. A numerical approximation of this PDF is provided in Fig. 6, where we plot a histogram of the plateau sizes. We have two clearly distinguishable regimes: (i) a power-law scaling $P(\tau) \sim \tau^{-\beta}$, with $\beta = -0.469 \pm 0.016$, valid for small plateaus; and (ii) an exponential (or fat) tail $P(\tau) \sim e^{\gamma \tau}$, with $\gamma = -0.00067 \pm 0.00001$ for large plateaus. The existence of these two scalings is a characteristic feature of on-off intermittency with noise, the existent crossover between them being related to the noise level. In our case of a spatially extended system the noisy effect is provided by the irregular forcing of the spatial modes excited in the Fourier space.

This work was made possible by partial financial support of CNPq, CAPES, Fundação Araucária, and RNF-CNEN (Brazilian Fusion Network).